Partially asymmetric exclusion process with open boundaries

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Exclusive diffusion on a one-dimensional lattice is studied. In the model, particles hop stochastically in both directions but with different rates. At the ends of the lattice, particles are injected and removed. The exact stationary probability measure is represented in the form of a matrix product, as a generalization of the solution given by Derrida *et al.* [J. Phys. A **26**, 1493 (1993)] for the fully asymmetric process. The phase diagram of the current on the infinite lattice is obtained. Analytic expressions for the current in the different phases are derived. The model is equivalent to an XXZ-Heisenberg chain with a certain type of boundary terms, the ground state of which corresponds to the stationary solution of the master equation.

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I. INTRODUCTION

The one-dimensional stochastic exclusion process is of interest for several reasons. Besides being the simplest example for diffusion of interacting particles [2, 3] it is closely related to various other phenomena such as interface growth [4, 5], the dynamics of shocks [6, 7] or of directed polymers [4], as well as freeway traffic [8–10]. Furthermore it can be mapped onto vertex models [11] or quantum spin chains [12–14]. Although the models are fairly simple, only few exact results are known [1–7,16–24].

Of particular interest is the asymmetric exclusion process with open boundaries as an example for a driven diffusive system coupled to its environment. For the fully asymmetric process where particles hop only in one direction, the exact stationary state is known. The problem was solved by Derrida, Domany, and Mukamel [18] for special choices of the system parameters and generalized by Schütz and Domany [20] and by Derrida *et al.* [1]. Several phase transitions were found for this model. The partially asymmetric process where particles are allowed to hop into both directions, but with different rates, is a natural generalization of the fully asymmetric process. Some exact results are known for periodic boundary conditions [21] as well as for a closed chain [24, 14]. In both cases characteristic time scales can be determined. In case of open boundaries an algebraic representation of the stationary solution was proposed in [1] for particular choices of the input and output rates.

The aim of this paper is to study the partially asymmetric process with open boundaries. The method used here is the algebraic ansatz introduced in [1]. Our main result is the stationary solution of the master equation. From this, expressions for the current on a large lattice are deduced [see eqs. (4.12)–(4.14)]. As for the fully asymmetric process, three phases are encountered (see Fig. 1 in Sec. IV), assuming that the direction of the drift is fixed. It turns out that these phases reduce to the ones of the fully asymmetric process in the appropriate limit.

The paper is organized as follows. In Sec. II the model is defined. Then, in Sec. III the stationary solution is introduced and two representations are discussed. The stationary current on a large lattice and its phase diagram are studied in Sec. IV. Section V concludes with some remarks on the density profile, mean field results, the algebraic structure of the solution, and the ground state of the corresponding XXZ chain.

II. MODEL

We consider a one-dimensional lattice of length L. Each lattice site can be occupied by one particle or can be empty. Hence the state of the system is defined by a set of occupation numbers $\tau_1, ..., \tau_L$ while $\tau_i = 1$ ($\tau_i = 0$) means site i is occupied (free). The particles are assumed to move stochastically on the lattice. With rate p they hop to their right if their nearest neighbor site on the right is empty and with a rate q they hop to the left if their left neighbor site is empty. Particles are injected at the left (right) boundary with a rate α (δ) and removed on the left (right) with a rate γ (β). The dynamics is supposed to be sequential, i.e., only one particle can hop at a time. Thus if the system has the configuration $\tau_1(t), ..., \tau_L(t)$ at time t it will change to the following:

for $1 < i < L$,

$$\tau_i(t+dt) = \begin{cases} 1 & \text{with probability } x_i = \tau_i(t) + [\{p\tau_{i-1}(t) + q\tau_{i+1}(t)\}\{1-\tau_i(t)\} \\ & \qquad -p\tau_i(t)\{1-\tau_{i+1}(t)\} - q\tau_i(t)\{1-\tau_{i-1}(t)\}]dt \\ 0 & \text{with probability } 1 - x_i \,; \end{cases} \tag{2.1}$$

for $i = 1$,

$$\tau_1(t+dt) = \begin{cases} 1 & \text{with probability } x_1 = \tau_1(t) + (\{\alpha + q\tau_2(t)\}\{1-\tau_1(t)\} - \tau_1(t)\{p[1-\tau_2(t)]\} + \gamma\})dt \\ 0 & \text{with probability } 1 - x_1 \ ; \end{cases} \quad (2.2)$$

and for $i = L$,

$$\tau_L(t+dt) = \begin{cases} 1 & \text{with probability } x_L = \tau_L(t) + (\{\delta + p\tau_{L-1}(t)\}\{1-\tau_L(t)\} - \tau_L(t)\{q[1-\tau_{L-1}(t)] + \beta\})dt \\ 0 & \text{with probability } 1 - x_L \ . \end{cases} \quad (2.3)$$

This defines a master equation for the probability distribution, which we may write as

$$\partial_t P(\tau_1, ..., \tau_L, t) = HP(\tau_1, ..., \tau_L, t) \ , \quad (2.4)$$

with

$$H = h_1' + \sum_{i=1}^{L-1} h_{i,i+1} + h_L' \ , \quad (2.5)$$

where the operator h_1' (h_L') describes the change of the probability by means of particle input (and output) at the left (right) boundary and $h_{i,i+1}$ gives the impact of the jumps in the bulk. The explicit form of the operators depends on the representation we choose.

The model exhibits two symmetries. It is invariant under the following exchanges:

$$\tau_i \leftrightarrow \tau_{L+1-i} \ , \quad p \leftrightarrow q \ , \quad \alpha \leftrightarrow \delta \ , \quad \beta \leftrightarrow \gamma \ , \quad (2.6)$$

or

$$\tau_i \leftrightarrow 1 - \tau_i \ , \quad p \leftrightarrow q \ , \quad \alpha \leftrightarrow \gamma \ , \quad \beta \leftrightarrow \delta \ . \quad (2.7)$$

This enables us to restrict our study to the case $p < q$ while results for the other part of the parameter space are obtained by exploiting one of the the above symmetries. Furthermore we may restrict our study to $p+q=1$. Any other value amounts to a rescaling of time only.

In the next section we are going to derive a stationary solution of the master equation (2.4).

Here let us make some remarks on the relation of the stochastic model to a quantum spin chain. We map the master equation on a imaginary time Schrödinger equation (see, e.g., [13, 23, 24]). In a basis defined by the vectors $|\tau_1, \tau_2, ..., \tau_L\rangle$ the Hamiltonian then reads:

$$\begin{aligned} H = -\sqrt{pq}\sum_{j=1}^{L-1}\Big\{ & \bar{q}^{-1}[c_j^\dagger c_{j+1} - (1-n_j)n_{j+1}] \\ & + \bar{q}[c_{j+1}^\dagger c_j - n_j(1-n_{j+1})]\Big\} \\ & -\alpha(c_1^\dagger - 1 + n_1) - \gamma(c_1 - n_1) \\ & -\delta(c_L^\dagger - 1 + n_L) - \beta(c_L - n_L) \ , \end{aligned} \quad (2.8)$$

with

$$\bar{q} = \sqrt{\frac{p}{q}} \ . \quad (2.9)$$

Here the operators $c_j^\dagger$ and c_j create and annihilate particles at site j. They obey spin commutation relations and may be written in terms of Pauli matrices: $c_j = (\sigma_j^x + i\sigma_j^y)/2$, $c_j^\dagger = (\sigma_j^x - i\sigma_j^y)/2$, and $n_j = (1-\sigma_j^z)/2$. Using the operator [15]

$$V = \bar{q}^{-\sum_{j=1}^{L} j n_j} \ , \quad (2.10)$$

the Hamiltonian (2.8) can be transformed into

$$H' = VHV^{-1} \ , \quad (2.11)$$

$$\begin{aligned} H' = -\frac{1}{2}\sqrt{pq}\sum_{j=1}^{L-1}\Big[& \sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y \\ & + \frac{1}{2}(\bar{q}+\bar{q}^{-1})\sigma_j^z\sigma_{j+1}^z - \frac{1}{2}(\bar{q}+\bar{q}^{-1})\Big] \\ & -A_1^+\sigma_1^x - iA_1^-\sigma_1^y - B_1\sigma_1^z - A_L^+\sigma_L^x - iA_L^-\sigma_L^y \\ & -B_L\sigma_L^z + \frac{1}{2}(\alpha+\beta+\gamma+\delta) \ , \end{aligned} \quad (2.12)$$

with

$$A_1^\pm = \frac{1}{2}(\gamma\bar{q} \pm \alpha\bar{q}^{-1}) \ ; B_1 = \frac{1}{2}(\gamma - \alpha) + \frac{1}{4}(\bar{q} - \bar{q}^{-1}) \quad (2.13)$$

and

$$A_L^\pm = \frac{1}{2}(\beta\bar{q}^L \pm \delta\bar{q}^{-L}) \ ; B_L = \frac{1}{2}(\beta - \delta) - \frac{1}{4}(\bar{q} - \bar{q}^{-1}) \ , \quad (2.14)$$

which is a spin-1/2 XXZ Hamiltionian with a certain class of boundary terms.

III. STATIONARY SOLUTION

An algebraic ansatz for the stationary measure of the fully asymmetric process was proposed in [1]. Proceeding in the same way we assume the system to have a stationary probability distribution which can be written as

$$P_L(\tau_1, ..., \tau_L) = \left\langle 0 \left| \prod_{i=1}^{L} [\tau_i D + (1-\tau_i)A] \right| 0 \right\rangle Z_L^{-1} \ , \quad (3.1)$$

where

$$Z_L = \langle 0|C^L|0\rangle \ , C = D + A \ , \quad (3.2)$$

with some matrices D and A and vectors $\langle 0|$ and $|0\rangle$. The following algebra was shown in [1] to give a stationary probability distribution for a lattice with up to three sites:

$$pDA - qAD = D + A \ , \quad (3.3)$$

$$(\beta D - \delta A)|0\rangle = |0\rangle \ , \quad (3.4)$$

$$\langle 0|(\alpha A - \gamma D) = \langle 0| \ . \quad (3.5)$$

This algebra defines a stationary solution for a lattice of any number of sites, which can be shown as follows. Consider the action of the operator $h_{i,i+1}$ from (2.5) on the probability of a certain state on a lattice of $L+1$ sites:

$$h_{i,i+1}P_{L+1}(\ldots\tau_{i-1}10\tau_{i+2}\ldots)$$
$$=-pP_{L+1}(\ldots\tau_{i-1}10\tau_{i+2}\ldots)+qP_{L+1}(\ldots\tau_{i-1}01\tau_{i+2}\ldots)\ .$$

Using (3.3) results in

$$h_{i,i+1}P_{L+1}(\ldots\tau_{i-1}10\tau_{i+2}\ldots)$$
$$=-k\{P_L(\ldots\tau_{i-1}0\tau_{i+2}\ldots)+P_L(\ldots\tau_{i-1}1\tau_{i+2}\ldots)\} \quad (3.6)$$

where $k=Z_L/Z_{L+1}$ and similarly

$$h_{i,i+1}P_{L+1}(\ldots\tau_{i-1}01\tau_{i+2}\ldots)$$
$$=k\{P_L(\ldots\tau_{i-1}0\tau_{i+2}\ldots)+P_L(\ldots\tau_{i-1}1\tau_{i+2}\ldots)\}\ . \quad (3.7)$$

Obviously

$$h_{i,i+1}P_{L+1}(\ldots\tau_{i-1}00\tau_{i+2}\ldots)=0 \quad (3.8)$$

and

$$h_{i,i+1}P_{L+1}(\ldots\tau_{i-1}11\tau_{i+2}\ldots)=0\ . \quad (3.9)$$

Using (3.4) and (3.5) we find

$$h'_1P_{L+1}(1\tau_2\ldots)=kP_L(\tau_2\ldots)\ , \quad (3.10)$$
$$h'_1P_{L+1}(0\tau_2\ldots)=-kP_L(\tau_2\ldots)\ , \quad (3.11)$$
$$h'_LP_{L+1}(\ldots\tau_L1)=-kP_L(\ldots\tau_L)\ , \quad (3.12)$$
$$h'_LP_{L+1}(\ldots\tau_L0)=kP_L(\ldots\tau_L)\ , \quad (3.13)$$

Equations (3.6)–(3.13) explicitly describe the action of the local parts of the Hamiltonian. Applying the full Hamiltonian $H=h'_1+\sum_{i=1}^{L}h_{i,i+1}+h'_L$ to the probability of any configuration on a lattice of length $L+1$, all occurring terms add up to zero. Consequently the algebra (3.3)–(3.5) defines a stationary probability distribution.

Let us remark that for periodic boundary conditions the stationary solution can be written as $P_L[\tau_1,\ldots,\tau_L]\propto$ Tr $\prod_{i=1}^{L}[\tau_iD+(1-\tau_i)A]$ with the algebra (3.3). This can be proved in the same way as above.

The algebra (3.3)–(3.5) in the above given form does not allow for explicit calculation of probabilities. However, the following relations, equivalent to Eqs. (3.3)–(3.5), define rules to compute $P_L(\tau_1\ldots\tau_L)$:

$$^{(1)}\langle k|=\langle 0|D^k\ , \quad (3.14)$$
$$^{(1)}\langle k|A=\sum_{i=0}^{k+1}a_{ki}\ ^{(1)}\langle i|\ , \quad (3.15)$$
$$^{(1)}\langle k|0\rangle=s_k\ \ \forall k=0,1,2,\ldots \quad (3.16)$$

with

$$a_{k0}=p^{-k}\alpha^{-1}\quad \forall k\geq 0\ , \quad (3.17)$$

$$a_{ki}=\alpha^{-1}\binom{k}{i}p^{-k}q^i\left\{\gamma q^{-1}\frac{i}{k-i+1}+1\right\}+\sum_{\nu=0}^{i-1}\binom{k-i+\nu}{i-1}p^{-k+i-\nu-1}q^{\nu}\quad \forall k\geq 0\ ,$$
$$0<i<k+1 \quad (3.18)$$

$$a_{kk+1}=p^{-k}q^k\alpha^{-1}\gamma\quad \forall k\geq 0\ , \quad (3.19)$$

and

$$\beta s_{k+1}=\delta\sum_{i=0}^{k+1}a_{ki}s_i+s_k\ . \quad (3.20)$$

Here we assumed $\alpha>0$.

Equation (3.14) is the definition of the vectors $^{(1)}\langle k|$. Equations (3.15) and (3.17)–(3.18) are consequences of Eq. (3.3) for the product DA and Eq. (3.5). Equations (3.16) and (3.20) follow from (3.4). A detailed proof of the equivalence between the algebra (3.3)–(3.5) and the rules (3.14)–(3.20) is given in Appendix A.

Note that in the case of totally asymmetric diffusion ($p=1$ and $q=\gamma=\delta=0$) studied in [1, 20] Eqs. (3.14)–(3.20) simplify to

$$^{(1)}\langle k|=\langle 0|D^k\ , \quad (3.21)$$
$$^{(1)}\langle k|A=\sum_{i=1}^{k}\ ^{(1)}\langle i|+\alpha^{-1}\langle 0|\ , \quad (3.22)$$
$$^{(1)}\langle k|0\rangle=\beta^{-k}\quad \forall k=0,1,2,\ldots\ . \quad (3.23)$$

Another similar representation of the algebra (3.3)–(3.5) can be found starting from $|k\rangle=A^k|0\rangle$ and proceeding in the same way as above.

The stationary probability distribution is given by Eq. (3.1). In order to find the probability of a certain configuration we have to calculate expressions such as $\langle 0|DADDDAA\ldots|0\rangle$, which reduce to linear combinations of the scalar products s_k after application of rules (3.14)–(3.19). Fixing s_0, say, to $s_0=1$, the latter quantities can be computed from the recursion (3.20). Hence Eqs. (3.1) and (3.2) combined with Eqs. (3.14)–(3.20) give a representation of the stationary probability distribution and enable us to calculate the probability of any configuration as well as any kind of averaged quantity. They may be used for numerical calculations. The dimension of the representation is $L+1$.

Here we introduce another representation of the algebra (3.3)–(3.5) which is more convenient for the large lattice approximations done in Sec. IV. Let us start with defining operators F and $F^\dagger$ by

$$D=\frac{1}{p-q}(F+1)\ , \quad (3.24)$$
$$A=\frac{1}{p-q}(F^\dagger+1)\ . \quad (3.25)$$

The relation (3.3) reads, in terms of F and $F^\dagger$,

$$F^\dagger F - \bar{q}^2 F F^\dagger = 1 - \bar{q}^2 \quad (\bar{q} = \sqrt{p/q} > 1) \ . \tag{3.26}$$

Operators commuting as above are known to be related to creation and annihilation operators of a q-deformed harmonic oscillator [25–27]. The latter operators may be defined as $a^\dagger = (\bar{q} - \bar{q}^{-1})^{-1/2} \bar{q}^{N/2} F^\dagger$ and $a = (\bar{q} - \bar{q}^{-1})^{-1/2} F \bar{q}^{N/2}$, where N is a particle number operator [25].

We choose a representation of (3.26) as [25]

$$F^\dagger |k\rangle^{(2)} = \{k+1\}_{\bar{q}}^{1/2} |k+1\rangle^{(2)} \ , \tag{3.27}$$

$$F|k\rangle^{(2)} = \{k\}_{\bar{q}}^{1/2} |k-1\rangle^{(2)} \ , \tag{3.28}$$

with

$$x_{\bar{q}} = 1 - \bar{q}^{-2x} \quad \text{for any } x > 0 \ , \tag{3.29}$$

where the $|k\rangle^{(2)}$, with $k = 0, 1, 2, \ldots$, form an orthogonal basis in an infinite-dimensional Hilbert space. $F^\dagger$ is adjoint to F. Equations (3.24), (3.25), and (3.27)–(3.29) specify the action of the operators D and A in this basis. Note that up to a factor the q-oscillator operators $a^\dagger$ and a act like $F^\dagger$ and F here. The consistence of the representation with Eqs. (3.26) is obvious.

For the computation of probabilities according to (3.1) we need furthermore to project the vectors $\langle 0|$ and $|0\rangle$ on the basis vectors in a way that the relations (3.4) and (3.5) representing the boundary conditions are ensured. Let us write

$$|0\rangle = \sum_{i=0}^{\infty} r_i \ |i\rangle^{(2)} \ , \tag{3.30}$$

$$\langle 0| = \sum_{i=0}^{\infty} l_i \ {}^{(2)}\langle i| \ . \tag{3.31}$$

Equation (3.4) reads, in terms of $F^\dagger$ and F,

$$[\beta F - \delta F^\dagger + \beta - \delta - p + q]|0\rangle = 0 \ , \tag{3.32}$$

which results after using Eq. (3.30) as well as (3.27) and (3.28) in

$$0 = \beta\{k+1\}_{\bar{q}}^{1/2} r_{k+1} + (\beta - \delta - p + q) r_k - \delta \{k\}_{\bar{q}}^{1/2} r_{k-1} \ , \tag{3.33}$$

where

$$r_0 = 1, \quad r_{-1} = 0 \ . \tag{3.34}$$

From Equation (3.5), in the same way we find, for the left vector,

$$0 = \alpha\{k+1\}_{\bar{q}}^{1/2} l_{k+1} + (\alpha - \gamma - p + q) l_k - \gamma \{k\}_{\bar{q}}^{1/2} l_{k-1} \ , \tag{3.35}$$

where

$$l_0 = 1, \quad l_{-1} = 0 \ . \tag{3.36}$$

Equation (3.35) can be obtained from Eq. (3.33) by replacing r_k by l_k, β by α, and δ by γ. Equations (3.30) and (3.31) give a representation of the boundary vectors in the above defined basis. Together with Eq. (3.24), (3.25), and (3.27)–(3.29), which specify the action of D and A, we may calculate any probability supposing we have computed the coefficient r_k and l_k by means of the recursions (3.33) and (3.35). The procedure is not very convenient for numerical examples since we have to deal with an infinite-dimensional representation. However large lattice approximations are seen more easily here than in the first representation. This is because of the impact of the left boundary, of the right boundary, and of the bulk enter in the left vector, in the right vector, and in the matrices, respectively.

The physical quantity we are to analyze in detail is the current j, which in the stationary state is independent of the position. It can be written in a convenient way as

$$j = \langle 0|C^{L-2}(pDA - qAD)C|0\rangle Z_L^{-1} = Z_{L-1} Z_L^{-1} \tag{3.37}$$

because of Eqs. (3.1)–(3.3). Therefore we have to calculate $Z_L = \langle 0|C^L|0\rangle$. Let us define the coefficients $c_{ik}^L = {}^{(2)}\langle i|C^L|k\rangle^{(2)}$ as the matrix elements of the operator C^L. Applying $C = (F^\dagger + F + 2)/(p-q)$ from the left to

$$C^L |k\rangle^{(2)} = \sum_{i=0}^{\infty} c_{ik}^L |i\rangle^{(2)} \tag{3.38}$$

and using the rules (3.27) and (3.28) we get a recursion relation

$$c_{ik}^{L+1} = \frac{1}{p-q} [\ \{i\}_{\bar{q}}^{1/2} c_{i-1k}^L + 2 c_{ik}^L + \{i+1\}_{\bar{q}}^{1/2} c_{i+1k}^L \ ] \ , \tag{3.39}$$

with

$$c_{ik}^0 = \delta_{i,k} \ , \quad c_{ik}^L = 0$$

$$\text{for } i > k + L \text{ or } i < \max\{k - L, 0\} \ . \tag{3.40}$$

Using the decompositions of the boundary vectors (3.30) and (3.31) as well as Eq. (3.38) we find the following expression for $Z_L = \langle 0|C^L|0\rangle$:

$$Z_L = \sum_{k,i=0}^{\infty} l_i c_{ik}^L r_k \ . \tag{3.41}$$

After solving the recursions for l_k, r_i, and c_{ki}^L the above equation allows for the calculation of Z_L and hence of the current. In the next section we are going to do this for a large lattice.

IV. THE STATIONARY CURRENT FOR A LARGE LATTICE

The following approximation is derived in Appendix B:

$$r_k \propto \begin{cases} [\kappa_+(\beta,\delta)]^k & \text{for } \kappa_+(\beta,\delta) > 1, \ k \gg 1 \\ [\kappa_-(\beta,\delta)]^k & \text{for } \kappa_+(\beta,\delta) < 1, \ k \gg 1 \end{cases} \tag{4.1}$$

with

$$\kappa_\pm(\beta,\delta)=\frac{1}{2\beta}[-\beta+\delta+p-q \pm\sqrt{(-\beta+\delta+p-q)^2+4\beta\delta}\,] \ . \tag{4.2}$$

For all choices of parameters $|\kappa_-(\beta,\delta)| < 1$. That means for $\kappa_+(\beta,\delta) < 1$ the coefficient r_k decreases exponentially with k whereas it increases exponentially for $\kappa_+(\beta,\delta) > 1$. Note that the condition $\kappa(\beta,\delta) > 1$ is equivalent to $\beta-\delta < \frac{p-q}{2}$.

Similarly we find, for the coefficients l_k,

$$l_k \propto \begin{cases} [\kappa_+(\alpha,\gamma)]^k & \text{for } \kappa_+(\alpha,\gamma) > 1,\ k \gg 1 \\ [\kappa_-(\alpha,\gamma)]^k & \text{for } \kappa_+(\alpha,\gamma) < 1\ ,\ k \gg 1 \end{cases} \tag{4.3}$$

with

$$\kappa_\pm(\alpha,\gamma)=\frac{1}{2\alpha}[-\alpha+\gamma+p-q \pm\sqrt{(-\alpha+\gamma+p-q)^2+4\alpha\gamma}\,] \tag{4.4}$$

and $|\kappa_-(\alpha,\gamma)| < 1$. That means for $\alpha-\gamma > \frac{p-q}{2}$, where $\kappa_+(\alpha,\gamma) < 1$, the coefficient r_k decreases exponentially with k whereas it increases exponentially for $\kappa_+(\alpha,\gamma) > 1$.

The coefficients c^L_{ik} are found to obey a simple relation

$$c^L_{ik}/c^{L-1}_{ik} \approx \left(\frac{4}{p-q}\right) \quad \text{for } k \ll L\ , i \ll L\ ,\ \gg 1\ . \tag{4.5}$$

This can be shown analytically for $\overline{q} \to \infty$, where a recursion of the same type as for the c^L_{ik} was solved in [1]. We did not succeed in deriving Eq. (4.5) for general $\overline{q}$, but the recursion (3.39) can be solved numerically. For all choices of $\overline{q}$ the above relation turns out to be correct. This result is plausible since, after removing the factor $1/(p-q)$, the recursion is similar to that for the simpler special case. Only additional factors $\{k\}^{1/2}_{\overline{q}}$ occur, which are approximately 1 for almost all k.

We use this approximation for the calculation of Z_L by means of Eq. (3.41). Different cases have to be distinguished.

The case $\kappa_+(\beta,\delta) < 1$ *and* $\kappa_+(\alpha,\gamma) < 1$. Here both of the coefficients r_i,l_i fall exponentially with i. Hence the main contribution to Z_L is given by the terms with small i,k in the sum. For small i,k the approximation (4.5) for the c^L_{ik} can be used. The region of validity of (4.5) increases with L while the descent of r_i and l_i is independent of L. Therefore we find, for $L \gg 1$, $Z_L/Z_{L-1} \approx \frac{4}{p-q}$ and, for the current,

$$j \approx \frac{p-q}{4}\ . \tag{4.6}$$

The case $\kappa_+(\beta,\delta) > \kappa_+(\alpha,\gamma)$ *and* $\kappa_+(\beta,\delta) > 1$. Here the coefficient r_k increases exponentially with k. l_i may or may not increase with i. In any case l_i cannot increase faster than r_i. Suppose now l_i increases as well. Then the sum (3.41) does not exist since it extends over terms such as $[\kappa_+(\beta,\delta)\kappa_+(\alpha,\gamma)]^k$. In order to avoid this divergence we redefine $\langle 0|$ as $\langle 0| = \lim_{N\to\infty}\{\ \sum_{i=0}^N l_i\ {}^{(2)}\langle i|\ [\sum_{j=0}^N \lambda^j]^{-1}\ \}$ with some real λ. Choosing $\lambda > \kappa_+(\beta,\delta)\kappa_+(\alpha,\gamma)$ ensures convergence of the sum for Z_L. Consider now the product $\langle 0|C^{L-1}F|0\rangle$. Since $F|0\rangle = \sum_{k=0}^\infty \{k+1\}^{1/2}_{\underline{q}} r_{k+1}|k\rangle$ we may write

$$\langle 0|C^{L-1}F|0\rangle = \lim_{N\to\infty}\left\{ \sum_{i=0}^N\sum_{k=0}^\infty \{k+1\}^{1/2}_{\underline{q}} \times l_i c^{L-1}_{ik} r_{k+1}\left[\sum_{j=0}^N \lambda^j\right]^{-1}\right\} \approx \kappa_+(\beta,\delta)Z_{L-1}\ , \tag{4.7}$$

where we have used the fact that the main contribution to the k sum for any i stems from large k terms. Small k terms with large i contribute less to the whole sum than large k terms since r_k increases faster than l_i. For $k \gg 1$ we can approximate $r_k \propto [\kappa_+(\beta,\delta)]^k$ and $\{k+1\}^{1/2}_{\overline{q}} \approx 1$. In the case that l_i does not increase with i, the same argument applies.

Writing now C as $C = \delta^{-1}[\ -(\beta D-\delta A) + \frac{1}{p-q}(\beta+\delta) + \frac{1}{p-q}(\beta+\delta)F\ ]$ we get

$$\begin{aligned} Z_L &= \delta^{-1}\left\langle 0\middle| C^{L-1}\left\{-(\beta D-\delta A) + \frac{1}{p-q}(\beta+\delta)+\frac{1}{p-q}(\beta+\delta)F\right\}\middle|0\right\rangle \\ &= \delta^{-1}\frac{1}{p-q}\{(\beta+\delta-p+q)Z_{L-1} +(\beta+\delta)\langle 0|C^{L-1}F|0\rangle\}\ . \end{aligned} \tag{4.8}$$

Inserting Eq. (4.7) into Eq. (4.8) results in

$$Z_L \approx Z_{L-1}\delta^{-1}\frac{1}{p-q}\{(\beta+\delta-p+q)+(\beta+\delta)\kappa_+(\beta,\gamma)\}\ . \tag{4.9}$$

Finally, using Eq. (4.2) for $\kappa_+(\beta,\gamma)$ we find, for the current $j \approx Z_{L-1}/Z_L$,

$$j \approx \frac{1}{2(p-q)}\ \{\ (\beta-\delta)(p-q) - (\beta+\delta)^2 +(\beta+\delta)\sqrt{(\beta-\delta-p+q)^2+4\beta\delta}\ \}\ . \tag{4.10}$$

The case $\kappa_+(\alpha,\gamma) > \kappa_+(\beta,\delta)$ *and* $\kappa_+(\alpha,\gamma) > 1$. The current for this case can be derived simply from the above result by applying both symmetry operations (2.6) and (2.7). Since the current does not depend on the position on the lattice we just have to replace β by α and δ by γ in Eq. (4.10):

$$j \approx \frac{1}{2(p-q)}\ \{\ (\alpha-\gamma)(p-q) - (\alpha+\gamma)^2 +(\alpha+\gamma)\sqrt{(\alpha-\gamma-p+q)^2+4\alpha\gamma}\ \}\ . \tag{4.11}$$

In summary, assuming the above approximations to be exact for $L \to \infty$, we have found analytic expressions for the current j on an infinite lattice. The dependence of

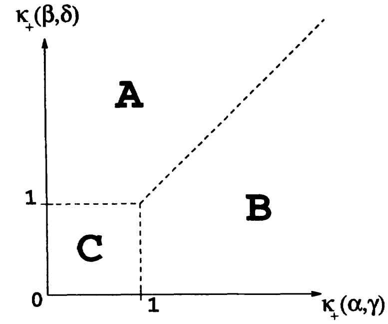

FIG. 1. Phase diagram for the current on a large lattice for $p > q$ in terms of $\kappa_+(\alpha,\gamma)$ and $\kappa_+(\beta,\delta)$, where $\kappa_+(x,y) = \frac{1}{2x}[-x+y+p-q+\sqrt{(-x+y+p-q)^2+4xy}\,]$. The phases are separated by the dashed lines.

j on the system parameters is different in three regions of the parameter space. Figure 1 shows the phase diagram. The phase separation lines are defined by means of the nonlinear expressions $\kappa_+(\beta,\delta)$ and $\kappa_+(\alpha,\gamma)$ [see Eqs. (4.2) and (4.4)]. In the fully asymmetric limit these functions reduce to $\kappa_+(\alpha,\gamma) = (1-\alpha)/\alpha$, $\kappa_+(\beta,\delta) = (1-\beta)/\beta$ for $\alpha,\beta < 1$ and the phase diagram is the one described in [1, 20].

The currents in the three phases are as follows:

Phase A. $\kappa_+(\beta,\delta) > \kappa_+(\alpha,\gamma)$ and $\kappa_+(\beta,\delta) > 1$

$$j = \frac{1}{2(p-q)}\ \{\ (\beta-\delta)(p-q) - (\beta+\delta)^2 + (\beta+\delta)\sqrt{(\beta-\delta-p+q)^2+4\beta\delta}\ \}\ . \qquad (4.12)$$

Note that the condition $\kappa_+(\beta,\delta) > 1$ is fulfilled if $\beta - \delta < \frac{p-q}{2}$. In the fully asymmetric limit, i.e., for $p = 1$, $q = \gamma = \delta = 0$, the high density phase described in [20, 1] is recovered.

Phase B. $\kappa_+(\alpha,\gamma) > \kappa_+(\beta,\delta)$ and $\kappa_+(\alpha,\gamma) > 1$:

$$j = \frac{1}{2(p-q)}\ \{\ (\alpha-\gamma)(p-q) - (\alpha+\gamma)^2 + (\alpha+\gamma)\sqrt{(\alpha-\gamma-p+q)^2+4\alpha\gamma}\ \}\ . \qquad (4.13)$$

The condition $\kappa_+(\alpha,\gamma) > 1$ is fulfilled if $\alpha-\gamma < (p-q)/2$. This phase corresponds to the low density phase in the fully asymmetric limit.

Phase C. $\kappa_+(\beta,\delta) < 1$ and $\kappa_+(\alpha,\gamma) < 1$:

$$j = \frac{p-q}{4}\ . \qquad (4.14)$$

In the fully asymmetric limit the maximum current phase is recovered.

These phases were found for $p > q$. For $p < q$ similar results are derived in a trivial manner using one of the symmetries (2.6) or (2.7).

Numerical calculations can be done most easily in the first representation of the algebra. They show that for typical choices of parameters (not to close to the phase lines) in the phases A or B the current has its value given above already for $L < 50$. In the phase C the convergence is much slower (about factor 10).

V. CONCLUSION

By fixing the preferred direction of the diffusion, three phases have been found for which the current on an infinite lattice obeys different equations. They correspond to the phases known for the fully asymmetric process which are recovered in the appropriate limit. The transition lines as well as the current are described by nonlinear functions of the system parameters. The full phase structure of the process may be richer taking into account transitions in the behavior of other quantities such as the density profile which have not been studied here. We argue that the situation here is similar to the one observed in the fully asymmetric model [20].

A mean field approximation can be applied in a way similar to what was done for the fully asymmetric process [18]. As in Ref. [18], it turns out that the obtained current and the phase structure are exact.

The density profile is expected to be similar to the one for the fully asymmetric process. As it can be seen by numerical calculations or by simple arguments using the first representation discussed in Sec. III, the density profile in phases A and B approaches a constant value from one of the boundaries. In phase C the density is constant in the bulk while it varies near both of the boundaries.

The stationary solution exhibits an interesting algebraic structure. The commutation relations (3.26) are related to the ones of creation and annihilation operators of a q-deformed oscillator. The latter ones can be use to construct the generators of the quantum group $U_q[SU(2)]$ [25], which is the symmetry group of asymmetric diffusion on a closed chain [24, 14]. On the other hand, the q-oscillator algebra can be obtained from the quantum group $U_q[SU(2)]$ as a large j limit [27].

Since the stochastic model can be mapped on a spin-1/2 XXZ chain its stationary solution corresponds to the ground state of this quantum system. Precisely speaking, the state $|\psi_0\rangle = \sum_{\{\tau_i\}} P_L(\tau_1,,...,\tau_L)|\tau_1,,...,\tau_L\rangle$, with $P_L(\tau_1,,...,\tau_L)$ given in Eq. (3.1), is the ground state of the Hamiltonian (2.8). Here $\tau_j = 0$ means spin up and $\tau_j = 1$ means spin down. The ground state of the Hamiltonian H' defined by Eqs. (2.12)–(2.14) is $V|\psi_0\rangle = \overline{q}^{\frac{1}{2}\sum_{j=1}^{L} j\sigma_j^z}|\psi_0\rangle$.

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APPENDIX A

Here we prove the equivalence between the algebras (3.3)–(3.5) and (3.14)–(3.20). Equation (3.14) is the definition of the vectors ${}^{(1)}\langle 0|$. Equation (3.15) reduces for $k = 0$ to Eq. (3.5). To show its equivalence to (3.3) and (3.5) for any k, we have to apply A to $\langle k|$ iteratively, i.e,

$$\langle 0|A=\alpha^{-1}\langle 0|+\alpha^{-1}\gamma\,{}^{(1)}\langle 1|\ ,$$
$${}^{(1)}\langle 1|A=\langle 0|DA$$
$$=p^{-1}\langle 0|(D+A+qAD)$$
$$=p^{-1}\{q\alpha^{-1}\gamma\,{}^{(1)}\langle 2|+(1+\alpha^{-1}\gamma+q\alpha^{-1})\times{}^{(1)}\langle 1|+\alpha^{-1}\langle 0|\}\ ,$$
$${}^{(1)}\langle 2|A={}^{(1)}\langle 1|DA=\cdots\ . \tag{A1}$$

From this we see

$${}^{(1)}\langle k|A=\sum_{i=0}^{k+1}a_{ki}\,{}^{(1)}\langle i| \tag{A2}$$

with

$$a_{k0}=p^{-k}\alpha^{-1}\quad\forall k\geq 0\ , \tag{A3}$$
$$a_{ki}=\alpha^{-1}\binom{k}{i}p^{-k}q^{i}\left\{\gamma q^{-1}\frac{i}{k-i+1}+1\right\}+p^{-k}a'_{ki}\quad\forall k\geq 0\ ,0<i<k+1\ , \tag{A4}$$
$$a_{kk+1}=p^{-k}q^{k}\alpha^{-1}\gamma\quad\forall k\geq 0\ . \tag{A5}$$

The coefficients a'_{ki} obey the following recursion relation:

$$a'_{ki}=pa'_{k-1i-1}+\binom{k-1}{i-1}q^{i-1}\quad\forall 0<i\leq k\ , \tag{A6}$$

which results in

$$a'_{ki}=\sum_{\nu=0}^{i-1}\binom{k-i+\nu}{\nu}q^{\nu}p^{i-\nu-1}\ . \tag{A7}$$

Inserting Eq. (A7) into (A4) results in the expression (3.18) for the coefficients c_{ki}, which prove the equivalence of Eqs. (3.3) and (3.5) to Eqs. (3.14), (3.15) and (3.17)–(3.18).

Multiplying ${}^{(1)}\langle k|$ from the left by Eq. (3.4) and using Eq. (3.16) gives

$${}^{(1)}\langle k|(\beta D-\delta A)|0\rangle=s_k\ . \tag{A8}$$

Using Eqs. (3.14) and (3.15) we find

$$\beta s_{k+1}=\delta\sum_{i=0}^{k+1}a_{ki}s_i+s_k\ , \tag{A9}$$

which is Eq. (3.20). This completes the proof of the equivalence of the algebras (3.14)–(3.20) on the one hand and (3.3)–(3.5) on the other.

APPENDIX B

In this appendix the large k approximation (4.1) and (4.2) for the components of the right boundary vector r_k is derived. For $k\gg 1$ the recursion relation (3.33) reads

$$0=\beta r_{k+1}+(\beta-\delta-p+q)r_k-\delta r_{k-1}\ . \tag{B1}$$

It is solved by the ansatz $r_k=b\kappa^k$, which results in

$$0=\beta\kappa^2+(\beta-\delta-p+q)\kappa-\delta\ . \tag{B2}$$

This quadratic equation for κ has two solutions:

$$\kappa_\pm(\beta,\delta)=\frac{1}{2\beta}[-\beta+\delta+p-q\pm\sqrt{(-\beta+\delta+p-q)^2+4\beta\delta}\,]\ . \tag{B3}$$

Hence we may approximate, for $k\gg 1$,

$$r_k=b_+[\kappa_+(\beta,\delta)]^k+b_-[\kappa_-(\beta,\delta)]^k\ . \tag{B4}$$

From Eq. (B3) we see that for $\beta-\delta<(p-q)/2$, $\kappa_+(\beta,\delta)>1$ whereas for all choices of parameters $\kappa_-(\beta,\delta)<1$. For $\kappa_+(\beta,\delta)>1$ the second decreasing term in Eq. (B4) is negligible. On the other hand, if $\beta-\delta>(p-q)/2$, i.e., $\kappa_+(\beta,\delta)<1$, we still find $|\kappa_-(\beta,\delta)|<1$ and $|\kappa_-(\beta,\delta)|>|\kappa_+(\beta,\delta)|$. Then the first term in Eq. (B4) becomes negligible in comparison to the second one. Consequently

$$r_k\propto\begin{cases}[\kappa_+(\beta,\delta)]^k & \text{for }\ \kappa_+(\beta,\delta)>1\ ,\quad k\gg 1\\ [\kappa_-(\beta,\delta)]^k & \text{for }\ \kappa_+(\beta,\delta)<1\ ,\quad k\gg 1\ .\end{cases} \tag{B5}$$

This is the large k approximation for the r_k. For the left boundary vector components l_k we find a similar expression since the recursion for l_k has the same structure as the one for r_k. Only β is replaced by α and γ is replaced by δ.

[1] B. Derrida, M.R. Evans, V. Hakim, and V. Pasquier, J. Phys. A **26**, 1493 (1993).
[2] T. Ligget, *Interacting Particle Systems* (Springer-Verlag, New York, 1985).
[3] F. Spitzer, Adv. Math. **5**, 246 (1970).
[4] J. Krug and H. Spohn, in *Solids far from Equilibrium*, edited by C. Godreche (Cambridge University Press, Cambridge, England, 1991).
[5] D. Kandel and D. Mukamel, Europhys. Lett. **20**, 325 (1992); D. Kandel, G. Gershinsky, and D. Mukamel, Phys. Scr. **T49**, 622 (1993)
[6] J. Krug, Phys. Rev. Lett. **67**, 1882 (1991).
[7] B. Derrida, S. A. Janowski, J. L. Lebowitz, and E. R. Speer, Europhys. Lett. **22**, 651 (1993).
[8] A. Schadschneider and M. Schreckenberg, J. Phys. A **26**, L679 (1993).
[9] K. Nagel and H. J. Herrmann, J. Phys. I **2**, 2221 (1993).
[10] T. Nagatani, J. Phys. A **26**, 6625 (1993).
[11] D. Kandel, E. Domany, and B. Nienhuis, J. Phys. A **23**, L755 (1990).
[12] P. Grassberger and M. Scheunert, Fortschr. Phys. **28**, 547 (1980).
[13] S. Sandow and S. Trimper, Europhys. Lett. **21**, 799 (1993).
[14] F. C. Alcaraz, M. Droz, M. Henkel, and V. Rittenberg, Ann. Phys. (N.Y.) **230**, 250 (1994).
[15] M. Henkel and G. Schütz, Physica A **206**, 187 (1994).
[16] J. L. Lebowitz, E. Presuutti, and H. Spohn, J. Stat. Phys.

51, 841 (1988).
[17] A. De Masi and P. Ferrari, J. Stat. Phys. **58**, 81 (1984).
[18] B. Derrida, E. Domany, and D. Mukamel, J. Stat. Phys. **69**, 667 (1992).
[19] B. Derrida, M. R. Evans, and D. Mukamel, J. Phys. A **26**, 4911, (1993).
[20] G. Schütz and E. Domany, J. Stat. Phys. **72**, 277 (1993).
[21] L.-H. Gwa and H. Spohn, Phys. Rev. Lett. **68**, 725 (1992); **46**, 844 (1992).
[22] G. Schütz, J. Stat. Phys. **71**, 471, (1993); Phys. Rev. E **47**, 4265 (1993).
[23] G. Schütz and S. Sandow, Phys. Rev. E **49**, 2726 (1994).
[24] S. Sandow and G. Schütz, Europhys. Lett. **26**, 7 (1994).
[25] A. J. Macfarlane, J. Phys. A **22**, 4580 (1989).
[26] L. C. Biedenharn, J. Phys. A **22**, L873 (1989).
[27] P. P. Kulish, Theor. Math. Phys. (USSR) **86**, 108 (1991).